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Some remarks on compact Einstein warped products

by

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Abstract. We consider a compact Einstein warped product $M = B \times_f F$ with $Ric_M = \lambda g_m$ where B is compact, connected, orientable, $dim\, B = m \geq 3$, F is compact, Einstein with $Ric_F = \mu g_F$, $dim\, F = k \geq 2$ and $f: B \longrightarrow (0, \infty)$, $f \in \mathcal{C}^\infty(B)$ a nonconstant warping function. In this article we study some upper bounds for some integrals involving the gradient and the Laplacian of the warping function f.

Keywords. compact Einstein space, warping function, gradient, Laplacian

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1. Introduction

The notion of warped product manifold was introduced in ([2]) where it served to give new examples of Riemannian manifolds. These methods were used to construct Einstein metrics on non-compact complete manifolds. The question of given examples of compact Einstein warped products appeared in ([1]). To that question a lot of results were given. It was proven in ([9]) that a compact Einstein warped product should have strictly positive scalar curvature. It was proven in ([6]) that the fibre of a compact Einstein warped product should also have strictly positive scalar curvature. Other results on Einstein spaces can be found in ([5], [7], [8]) and generalizations of Einstein spaces can be found in ([10]).

We present the Ricci tensor of a warped product.

Proposition 1.1. *([9]) The Ricci curvature Ric_M of an warped product $M = B \times_f F$ satisfies:*

a). $Ric_M(X,Y) = Ric_B(X,Y) - \frac{k}{f} H^f(X,Y)$

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b). $Ric_M(X,V) = 0$

c). $Ric_M(V,W) = Ric_F(V,W) - g_M(V,W)\left[-\frac{\Delta f}{f} + \frac{k-1}{f^2} g_B(\nabla f, \nabla f)\right]$

for every horizontal vectors X, Y *and every vertical vectors* V, W *where* ∇f *denotes the gradient of* f *and* Δf *denotes the laplacian of* f *given by* $\Delta f = -Tr(H^f) = -div(\nabla f)$.

Thus the Einstein equations become:

Corollary 1.1. *([9]) The warped product* $M = B \times_f F$ *is an Einstein space with* $Ric_M = \lambda g_m$ *if and only if*

(1.1) $Ric_B = \lambda g_B + \frac{k}{f} H^f$

(1.2) (F, g_F) *is an Einstein space with* $Ric_F = \mu g_F$ *for a constant* $\mu \in \mathbb{R}$

(1.3) $-f\Delta f + (k-1)|\nabla f|^2 + \lambda f^2 = \mu$

Throughout this paper we will use the following well-known results:

Theorem 1.1. *([3], Divergence Theorem 1) Let* B *be a compact, connected, orientable Riemannian manifold and* dv *the volume form on* B*. Then every vector field* X *on* B *satisfies the equality*

$$\int_B div(X) dv = 0$$

Theorem 1.2. *([4], Cauchy-Schwarz integral inequality) Let* B *be a compact, connected, orientable Riemannian manifold and* dv *the volume form on* B*. Let* f *and* g *be two differentiable functions on* B*. Then we have*

$$\left(\int_B f^2\, dv\right)\left(\int_B g^2\, dv\right) \geq \left(\int_B fg dv\right)^2$$

2. Main results

The aim of this paper is to study some upper bounds for some integrals involving the gradient and the laplacian of the warping $f: B \longrightarrow (0, \infty)$ on a compact Einstein warped product. Let $p, q \in$

B be such that $f(p) = \max_{x \in B} f(x)$ and $f(q) = \min_{x \in B} f(x)$. According to ([7]) we have $\Delta f(p) > 0$, $\Delta f(q) < 0$ and according to ([8]) we have the following formulas for λ and μ

$$\lambda = \frac{f(p)\Delta f(p) - f(q)\Delta f(q)}{f^2(p) - f^2(q)} > 0$$

$$\mu = f(p)f(q)\left[\frac{f(q)\Delta f(p) - f(p)\Delta f(q)}{f^2(p) - f^2(q)}\right] = f(p)f(q)\left[\lambda - \frac{\Delta f(p) + \Delta f(q)}{f(p) + f(q)}\right] > 0$$

Now we state the main theorem.

Theorem 2.1. Let $M = B \times_f F$ *be a compact Einstein warped product with* $Ric_M = \lambda g_m$, $\lambda > 0$ *where* B *is compact, connected, orientable,* $dim\, B = m \geq 3$, F *is compact, Einstein with* $Ric_F = \mu g_F$, $\mu > 0$, $dim\, F = k \geq 2$ *and a nonconstant warping function* $f: B \longrightarrow (0, \infty)$, $f \in C^\infty(B)$. *Let* $p, q \in B$ *be such that* $f(p) = \max_{x \in B} f(x)$ *and* $f(q) = \min_{x \in B} f(x)$.

a). If $k \geq 3$ *then we have*

$$\int_B |\nabla f| dv \leq \sqrt{-\frac{f(q)\Delta f(q)}{k-2}}\, vol(B)$$

$$\int_B f^2\, dv \leq \frac{f(p)f(q)[f(q)\Delta f(p) - f(p)\Delta f(q)]}{f(p)\Delta f(p) - f(q)\Delta f(q)} vol(B)$$

$$\int_B f\Delta f dv \leq -\frac{(k-1)f(q)\Delta f(q)}{k-2} vol(B)$$

b). If $k = 2$ *then we have*

$$\int_B \frac{|\nabla f|}{f} dv \leq \sqrt{-\frac{\Delta f(q)}{2f(q)}}\, vol(B)$$

$$\int_B f^2\, dv = \frac{f(p)f(q)[f(q)\Delta f(p) - f(p)\Delta f(q)]}{f(p)\Delta f(p) - f(q)\Delta f(q)} vol(B)$$

$$\int_B \frac{\Delta f}{f} dv \leq -\frac{\Delta f(q)}{2f(q)} vol(B)$$

Proof. a). Let $k \geq 3$. Then from equation (1.3) we obtain

$$-f\Delta f + (k-1)|\nabla f|^2 + \lambda f^2 = \mu \Longrightarrow$$
$$div(f\nabla f) + (k-2)|\nabla f|^2 + \lambda f^2 = \mu \Longrightarrow$$

$$\int_B div(f\nabla f)\,dv + (k-2)\int_B |\nabla f|^2\,dv + \lambda\int_B f^2\,dv = \int_B \mu\,dv \Longrightarrow$$

$$(k-2)\int_B |\nabla f|^2\,dv = \int_B \mu\,dv - \lambda\int_B f^2\,dv =$$

$$\int_B [-f(p)\Delta f(p) + \lambda f^2(p)]\,dv - \lambda\int_B f^2\,dv =$$

$$-f(p)\Delta f(p)vol(B) + \lambda\int_B [f^2(p) - f^2]\,dv \leq$$

$$-f(p)\Delta f(p)vol(B) + \lambda\int_B [f^2(p) - f^2(q)]\,dv =$$
$$-f(p)\Delta f(p)vol(B) + \lambda[f^2(p) - f^2(q)]vol(B) =$$

$$-f(p)\Delta f(p)vol(B) + \left[\frac{f(p)\Delta f(p) - f(q)\Delta f(q)}{f^2(p) - f^2(q)}\right][f^2(p) - f^2(q)]vol(B) =$$

$$-f(q)\Delta f(q)vol(B)$$

Thus, we obtain

$$\int_B |\nabla f|^2\,dv \leq -\frac{f(q)\Delta f(q)}{k-2}vol(B)$$

Now from Theorem 1.2 we have

$$\left(\int_B |\nabla f|\,dv\right)^2 \leq \left(\int_B |\nabla f|^2\,dv\right)\left(\int_B 1\,dv\right) \leq -\frac{f(q)\Delta f(q)}{k-2}vol^2(B) \Longrightarrow$$

$$\int_B |\nabla f|\,dv \leq \sqrt{-\frac{f(q)\Delta f(q)}{k-2}vol(B)}$$

Moreover

$$(k-2)\int_B |\nabla f|^2\, dv = \int_B \mu\, dv - \lambda \int_B f^2\, dv \geq 0 \Longrightarrow$$

$$\int_B f^2\, dv \leq \frac{\mu}{\lambda} vol(B) = \frac{f(p)f(q)[f(q)\Delta f(p) - f(p)\Delta f(q)]}{f(p)\Delta f(p) - f(q)\Delta f(q)} vol(B)$$

From equation (1.3) we also have

$$-f\Delta f + (k-1)|\nabla f|^2 + \lambda f^2 = \mu \Longrightarrow$$

$$f\Delta f = (k-1)|\nabla f|^2 + \lambda f^2 - \mu \Longrightarrow$$

$$\int_B f\Delta f\, dv = (k-1)\int_B |\nabla f|^2\, dv + \lambda \int_B f^2\, dv - \int_B \mu\, dv \leq$$

$$-\frac{(k-1)f(q)\Delta f(q)}{k-2} vol(B) +$$

$$\left[\frac{f(p)\Delta f(p) - f(q)\Delta f(q)}{f^2(p) - f^2(q)}\right]\left\{\frac{f(p)f(q)[f(q)\Delta f(p) - f(p)\Delta f(q)]}{f(p)\Delta f(p) - f(q)\Delta f(q)}\right\} vol(B) -$$

$$f(p)f(q)\left[\frac{f(q)\Delta f(p) - f(p)\Delta f(q)}{f^2(p) - f^2(q)}\right] vol(B) = -\frac{(k-1)f(q)\Delta f(q)}{k-2} vol(B) \Longrightarrow$$

$$\int_B f\Delta f\, dv \leq -\frac{(k-1)f(q)\Delta f(q)}{k-2} vol(B)$$

We remark that the equality cases occur for a constant warping function f.

b). Let $k = 2$. Then from equation (1.3) we have

$$-f\Delta f + |\nabla f|^2 + \lambda f^2 = \mu \Longrightarrow$$

$$-\frac{\Delta f}{f} + \frac{|\nabla f|^2}{f^2} + \lambda = \frac{\mu}{f^2} \Longrightarrow$$

$$div\left(\frac{\nabla f}{f}\right)+2\frac{|\nabla f|^2}{f^2}+\lambda=\frac{\mu}{f^2}\Longrightarrow$$

$$2\int_B \frac{|\nabla f|^2}{f^2}dv=\mu\int_B \frac{1}{f^2}dv-\int_B \lambda\,dv=$$

$$\mu\int_B \frac{1}{f^2}dv-\int_B\left[\frac{\mu}{f^2(p)}+\frac{\Delta f(p)}{f(p)}\right]dv=$$

$$\mu\int_B\left[\frac{1}{f^2}-\frac{1}{f^2(p)}\right]dv-\frac{\Delta f(p)}{f(p)}vol(B)\le$$

$$\mu\int_B\left[\frac{1}{f^2(q)}-\frac{1}{f^2(p)}\right]dv-\frac{\Delta f(p)}{f(p)}vol(B)=$$

$$f(p)f(q)\left[\frac{f(q)\Delta f(p)-f(p)\Delta f(q)}{f^2(p)-f^2(q)}\right]\left[\frac{1}{f^2(q)}-\frac{1}{f^2(p)}\right]vol(B)-$$

$$\frac{\Delta f(p)}{f(p)}vol(B)=-\frac{\Delta f(q)}{f(q)}vol(B)\Longrightarrow$$

$$\int_B \frac{|\nabla f|^2}{f^2}dv\le-\frac{\Delta f(q)}{2f(q)}vol(B)$$

Now from Theorem 1.2 we obtain

$$\left(\int_B \frac{|\nabla f|}{f}dv\right)^2\le\left(\int_B \frac{|\nabla f|^2}{f^2}dv\right)\left(\int_B 1\,dv\right)\le-\frac{\Delta f(q)}{2f(q)}vol^2(B)\Longrightarrow$$

$$\int_B \frac{|\nabla f|}{f}dv\le\sqrt{-\frac{\Delta f(q)}{2f(q)}vol(B)}$$

We remark that the equality case occurs for a constant warping function f.

Moreover

$$-f\Delta f+|\nabla f|^2+\lambda f^2=\mu\Longrightarrow$$

$$div(f\nabla f)+\lambda f^2=\mu \Longrightarrow$$

$$\lambda\int_B f^2\,dv=\int_B \mu\,dv \Longrightarrow$$

$$\int_B f^2\,dv=\frac{\mu}{\lambda}vol(B) \Longrightarrow$$

$$\int_B f^2\,dv=\frac{f(p)f(q)[f(q)\Delta f(p)-f(p)\Delta f(q)]}{f(p)\Delta f(p)-f(q)\Delta f(q)}vol(B)$$

We remark that

$$vol^2(B)=\left(\int_B 1\,dv\right)^2=\left(\int_B f\cdot\frac{1}{f}dv\right)^2\leq\left(\int_B f^2\,dv\right)\left(\int_B \frac{1}{f^2}dv\right)\Longrightarrow$$

$$\int_B \frac{1}{f^2}dv\geq\frac{vol^2(B)}{\int_B f^2\,dv}=\frac{f(p)\Delta f(p)-f(q)\Delta f(q)}{f(p)f(q)[f(q)\Delta f(p)-f(p)\Delta f(q)]}vol(B)$$

From equation (1.3) we also have

$$-f\Delta f+|\nabla f|^2+\lambda f^2=\mu \Longrightarrow$$

$$-\frac{\Delta f}{f}+\frac{|\nabla f|^2}{f^2}+\lambda=\frac{\mu}{f^2}\Longrightarrow$$

$$\frac{\Delta f}{f}=\frac{|\nabla f|^2}{f^2}+\lambda-\frac{\mu}{f^2}\Longrightarrow$$

$$\int_B \frac{\Delta f}{f}dv=\int_B \frac{|\nabla f|^2}{f^2}dv+\int_B \lambda\,dv-\mu\int_B \frac{1}{f^2}dv\leq$$

$$\left[-\frac{\Delta f(q)}{2f(q)}\right]vol(B)+\left[\frac{f(p)\Delta f(p)-f(q)\Delta f(q)}{f^2(p)-f^2(q)}\right]vol(B)-$$

$$\left\{\frac{f(p)f(q)[f(q)\Delta f(p)-f(p)\Delta f(q)]}{f(p)\Delta f(p)-f(q)\Delta f(q)}\right\}\left\{\frac{f(p)\Delta f(p)-f(q)\Delta f(q)}{f(p)f(q)[f(q)\Delta f(p)-f(p)\Delta f(q)]}\right\}vol(B)\Longrightarrow$$

$$\int_B \frac{\Delta f}{f} dv \leq -\frac{\Delta f(q)}{2f(q)} vol(B)$$

We remark that the equality case occurs for a constant warping function f.

Corollary 2.1. Let $M = B \times_f F$ *be a compact Einstein warped product with* $Ric_M = \lambda g_m, \lambda > 0$ *where* B *is compact, connected, orientable,* $dim\, B = m \geq 3$, F *is compact, Einstein with* $Ric_F = \mu g_F, \mu > 0, dim\, F = k \geq 2$ *and a nonconstant warping function* $f: B \longrightarrow (0, \infty), f \in C^{\infty}(B)$. *Let* $p, q \in B$ *be such that* $f(p) = \max_{x \in B} f(x)$ *and* $f(q) = \min_{x \in B} f(x)$. *Moreoever, we suppose that* $\Delta f(p) + \Delta f(q) = 0$.

a). If $k \geq 3$ *then we have*

$$\int_B |\nabla f| dv \leq \sqrt{\frac{f(q)\Delta f(p)}{k-2}} vol(B)$$

$$\int_B f^2 \, dv \leq f(p)f(q)vol(B)$$

$$\int_B f\Delta f dv \leq \frac{(k-1)f(q)\Delta f(p)}{k-2} vol(B)$$

b). If $k = 2$ *then we have*

$$\int_B \frac{|\nabla f|}{f} dv \leq \sqrt{\frac{\Delta f(p)}{2f(q)}} vol(B)$$

$$\int_B f^2 \, dv = f(p)f(q)vol(B)$$

$$\int_B \frac{\Delta f}{f} dv \leq \frac{\Delta f(p)}{2f(q)} vol(B)$$

Proof. It follows directly from Theorem 2.1 by considering the equation $\Delta(p) = -\Delta(q)$.

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